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Projective Representations of Quantum Logics

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1. *Introduction*

Probably the most important problem in the quantum logic approach to general quantum mechanics is to find physically reasonable postulates for a quantum logic so that it may be represented by the logic of all closed subspaces of a Hilbert space. Practically all investigators in this field have at least considered this problem (cf. Birkhoff & von Neumann, 1936; Gudder, 1969, 1970; Gunson, 1967; Jauch, 1968; Mackey, 1963; MacLaren, 1964; Piron, 1964; Pool, 1968b; Varadarajan, 1968; Zierler, 1961). The best result known to this author is due to Piron (1964) (with the help of Araki & Varadarajan, 1968) and is based on the fundamental theorem of projective geometry. He is able to construct a Hilbert space projective representation for so-called projective logics where a *projective logic* L is an orthomodular complete atomic lattice which satisfies:

(i) if $a \neq 0$ in L is the supremum of a finite set of atoms then $[0,a]$ is a geometry of finite rank;
(ii) if x, $a \in L$, $a \neq 0$, $\neq 1$ and x is an atom, then there are atoms y, $z \in L$ such that $y \leqslant a$, $z \leqslant a'$ and $x \leqslant y \vee z$;
(iii) there is at least one $a \in L$ such that $4 \leqslant \dim(a) < \infty$.

In the author's opinion Axiom (i) is particularly unfortunate, since it requires that the lattice $[0,a]$ be modular, and there seems to be no physical justification for such an axiom.

In this paper the author suggests some axioms which he feels are physically more reasonable and which imply some of those given above. (In particular that L is a complete atomic lattice satisfying (i).) These new axioms deal with superpositions of states and the superposition principle.

2. *A Superposition Principle*

Let L be an orthocomplemented poset. That is, L is a partially ordered set with first and last elements 0, 1 respectively and a complementation $a \to a'$ satisfying (i) $a'' = a$, (ii) $a \vee a' = 1$, (iii) if $a \leqslant b$, then $b' \leqslant a'$. We also assume that if a_i is a sequence of mutually disjoint elements (i.e., $a_i \leqslant a_j'$, $i \neq j$), then $\vee a_i$ exists. A map m from L into the real unit interval $[0,1]$ which satisfies $m(1) = 1$ and $m(\vee a_i) = \sum m(a_i)$ if the a_i's are disjoint is a *state* on L. If m is a state which cannot be written in the form $m = cm_1 + (1-c)m_2$, where $0 < c < 1$ and m_1 and m_2 are distinct states, then m is called a *pure state*. We denote the set of states on L by M and the set of pure states on L by P. If $a \in L, m \in P$, define $P_a = \{m \in P: m(a) = 1\}, L_m = \{a \in L: m(a) = 1\}$. If $P_a \subseteq P_b$ implies $a \leqslant b$ and $L_{m_1} \subseteq L_{m_2}$ implies $m_1 = m_2$, we call (L, M) a *quantum logic*. We say that $a, b \in L$ are *compatible* if there are mutually disjoint elements a_1, b_1, c such that $a = a_1 \vee c$ and $b = b_1 \vee c$. In the sequel (L, M) will always denote a quantum logic.

If $S \subseteq M$, $a \in L$ we write $S(a) = \alpha$ if $m(a) = \alpha$ for all $m \in S$. If $S \subseteq M$, $m_0 \in M$, then m_0 is a *superposition of states in* S if $S(a) = 0$ implies $m_0(a) = 0$ for all $a \in L$. If $S \subseteq P$ we denote by S^- the set of all pure states which are superpositions of states in S, and we define $\mathscr{M} = \{S \subseteq P: S = S^-\}$. Under set inclusion $\mathscr{M}$ becomes a poset with first and last elements ϕ, P respectively. We say that the *superposition principle holds* in (L, M) if $\mathscr{M}$ is isomorphic to L, i.e., if these exists a one-one map from $\mathscr{M}$ onto L that preserves order. (This definition is due to Varadarajan, 1968.) It is shown by Gudder (1969) that $\mathscr{M}$ is a complete atomic lattice. We thus have the following theorem.

Theorem 2.1

If the superposition principle holds in (L, M) then L is a complete atomic lattice.

We also have a kind of converse to Theorem 2.1. If L_1, L_2 are two orthocomplemented posets we say they are isomorphic if there is an order and complementation preserving isomorphism from L_1 onto L_2.

Theorem 2.2

If (L, M) is a quantum logic and L is a complete lattice for which $m(a_\alpha) = 1$, $\alpha \in A$, implies $m(\wedge a_\alpha) = 1$, then $\mathscr{M}$ has an orthocomplementation and L and $\mathscr{M}$ are isomorphic. In particular the superposition principle holds in (L, M).

Proof: for $S \in \mathscr{M}$ let $a_S = \wedge\{a \in L: S(a) = 1\}$. Then $a_S \in L$ is the smallest element for which $S(a_S) = 1$. Define $S' = \{m \in P: m(a_S) = 0\}$. Then $S' \in \mathscr{M}$. Suppose $m(a_S) = 1$, $m \in P$. If $S(b) = 0$, then $S(b') = 1$, so $a_S \leqslant b'$. Hence $m(b') = 1$ and $m(b) = 0$. Since $S = S^-$ we have $m \in S$, and hence $m(a_S) = 1$ if and only if $m \in S$. We now show that $S \to a_S$ is an isomorphism of $\mathscr{M}$ onto L. Suppose S_1, $S_2 \in \mathscr{M}$ and $S_1 \neq S_2$. If $m \in S_1$ and $m \notin S_2$, then $m(a_{S_1}) = 1$, while $m(a_{S_2}) \neq 1$. Thus, $a_{S_1} \neq a_{S_2}$ and the map is one-one. Let $a \in L$ and $S = L_a \in \mathscr{M}$. If $S = \phi$, then $a = 0$ and $a = a_\phi$. If $S \neq \phi$ we claim that $a = a_S$. Certainly $a_S \leqslant a$. If $a_S \neq a$, then there is $m_0 \in P$ such that $m_0(a) = 1$ and $m_0(a_S) \neq 1$. But $m_0 \in S$, which is a contradiction. Now suppose $S_1 \subseteq S_2$. If $m \in P$ and $m(a_{S_1}) = 1$, then $m \in S_1$. Hence, $m \in S_2$ and $m(a_{S_2}) = 1$. Therefore, $a_{S_1} \leqslant a_{S_2}$. Conversely, suppose $a_{S_1} \leqslant a_{S_2}$. If $m \in S_1$, then $m(a_{S_1}) = 1$, so $m(a_{S_2}) = 1$. Therefore, $m \in S$ and $S_1 \subseteq S_2$. Finally, the following statements are equivalent: $m(a_S') = 1$, $m(a_S) = 0$, $m \in S'$, $m(a_{S'}) = 1$, for $m \in P$. Hence $a_S' = a_{S'}$.

The condition in this theorem that $m(a_\alpha)=1$ implies $m(\wedge a_\alpha)=1$ has been used by Jauch (1968) and Piron (1964) in their formulation of quantum mechanics.

3. *Modularity*

In the sequel we assume that (L, M) is a quantum logic in which the superposition principle holds. In the last section we showed that L is then a complete atomic lattice. Notice that under the isomorphism pure states in M correspond to atoms in L.

We say that $m \in P$ is a *minimal superposition* of $m_i \in P$, $i=1, \ldots, n$, if $m \in \{m_i : i=1, \ldots, n\}^-$ but $m \notin \{m_i : i \neq j\}^-$ for any $j=1, \ldots, n$. Our next postulate is called the *minimal superposition postulate:* if m is a minimal superposition of $m_1, \ldots, m_n$ and (I, J) is a partition of $\{1, \ldots, n\}$ (i.e $\{1, \ldots, n\} = I \cup J$; $I \cap J = \phi$; $I, J \neq \phi$), then $\{m, m_i : i \in I\}^- \cap \{m_j : j \in J\}^- \neq \phi$. Our only comment on the physical nature of this postulate is that it is intuitively fairly clear. Indeed, suppose $m \in \{m_i : i=1, \ldots, n\}^-$ is a minimal superposition and (I, J) is a partition of $\{1, \ldots, n\}$. If $\{m, m_i : i \in I\}^- \cap \{m_j : j \in J\}^- = \phi$, then superpositions of m, m_i, $i \in I$ are not in $\{m_j : j \in J\}^-$ and are thus 'independent' of m_j, $j \in J$. Thus m_j, $j \in J$ are not needed to describe superpositions of m, m_i, $i \in I$ so we would have $m \in \{m_i : i \in I\}^-$, which contradicts the minimality.

Notice that the minimal superposition postulate holds in the usual Hilbert space framework. Indeed, in this case m, m_i, $i=1, \ldots, n$ may be represented by unit vectors ϕ, ϕ_i, $i=1, \ldots, n$. If m is a minimal superposition of m_i, $i=1, \ldots, n$, we have

$$\phi = \sum_{i=1}^{n} c_i \phi_i$$

for non-zero complex numbers c_i and distinct vectors ϕ_i, $i=1, \ldots, n$. Then for any partition (I, J) of $\{1, \ldots, n\}$ we have

$$\phi - \sum_{i \in I} c_i \phi_i = \sum_{j \in J} c_j \phi_j$$

Normalizing the vector

$$\sum_{j \in J} c_j \phi_j$$

we get a pure state in $\{m, m_i : i \in I\}^- \cap \{m_j : j \in J\}^-$.

In the lattice L we say that an atom a is a *minimal superposition* of atoms $a_1, \ldots, a_n$ if

$$a < \bigvee_{i=1}^{n} a_i$$

but

$$a \nleq \bigvee_{i \neq j} a_i$$

for any $j = 1, \ldots, n$. The minimal superposition postulate is thus equivalent to the following statement: if a is a minimal superposition of $a_1, \ldots, a_n$, then for any partition (I, J) of $\{1, \ldots, n\}$ we have

$$\left(a \vee \bigvee_{i \in I} a_i\right) \wedge \left(\bigvee_{j \in J} a_j\right) \neq 0$$

The main result of this section is that the minimal superposition postulate on (L, M) implies $[0,a]$ is modular for any a of finite rank. Let us now demonstrate that the converse holds. That is, let $\mathscr{L}$ be a modular lattice with a first element 0 and let $a, a_1, \ldots, a_n$ be atoms in $\mathscr{L}$. Suppose

$$a \leqslant \bigvee_{i=1}^{n} a_i$$

is a minimal superposition. Now assume (I, J) is a partition of $\{1, \ldots, n\}$ and that

$$\left(a \vee \bigvee_{i \in I} a_i\right) \wedge \left(\bigvee_{j \in J} a_j\right) = 0$$

Then by the modular law we have

$$a \leqslant \left(a \vee \bigvee_{i \in I} a_i\right) \wedge \left(\bigvee_{j \in J} a_j \vee \bigvee_{i \in I} a_i\right) = \left[\left(a \vee \bigvee_{i \in I} a_i\right) \wedge \bigvee_{j \in J} a_j\right] \vee \bigvee_{i \in I} a_i = \bigvee_{i \in I} a_i$$

which is a contradiction. Hence

$$\left(a \vee \bigvee_{i \in I} a_i\right) \wedge \left(\bigvee_{j \in J} a_j\right) \neq 0$$

and the minimal superposition postulate holds.

In the sequel we assume that (L, M) is a quantum logic in which the superposition principle and the minimal superposition postulate hold. We say that atoms $a_i \in L$, $i = 1, \ldots, n$ are *independent* if $a_i \nleqslant \bigvee \{a_j : j \neq i\}$ for any $i = 1, \ldots, n$.

Lemma 3.1

Suppose $a_1, \ldots, a_n$ are independent atoms and b is an atom. If $a_1 \leqslant b \vee a_2 \vee \cdots \vee a_n$ then

$$b \leqslant \bigvee_{i=1}^{n} a_i$$

Proof: There is a minimal subset $I \subseteq \{2, \ldots, n\}$ such that

$$a_1 \leqslant b \vee \bigvee_{i \in I} a_i$$

Since this is a minimal superposition it follows from the minimal superposition postulate that

$$b \wedge \left(a \vee \bigvee_{i \in I} a_i\right) \neq 0$$

Since b is an atom

$$b < \bigvee_{i=1} a_i$$

Corollary 3.2

If $a_1, \ldots, a_n$ are independent atoms and b is an atom satisfying

$$b \nleq \bigvee_{i=1}^{n} a_i$$

then $\{b, a_i : i = 1, \ldots, n\}$ are independent. We say that a finite set of atoms $a_1, \ldots, a_n$ is a basis for $a \in L$ if $a_1, \ldots, a_n$ are independent and

$$a = \bigvee_{i=1}^{n} a_i$$

Lemma 3.3

Let $a_1, \ldots, a_r$ be a basis for $a \in L$. Let $b_1, \ldots, b_n$ be atoms in a and suppose $n > r$. Then $b_1, \ldots, b_n$ are not independent.

Proof: Suppose $b_1, \ldots, b_n$ are independent. Now

$$b_1 < \bigvee_{i=1} a_i$$

and there is a minimal subset $I_1 \subseteq \{1, \ldots, r\}$ such that

$$b_1 < \bigvee_{i \in I} a_i$$

Without loss of generality assume $1 \in I_1$. Then $b_1 \nleq \bigvee \{a_j : j \in I_1 - \{1\}\}$ and by Corollary 3.2, $\{b, a_i : i \in I_1 - \{1\}\}$ are independent. Applying Lemma 3.1 we have $a_1 \leqslant b_1 \vee \bigvee \{a_j : j \in I_1 - \{1\}\}$ and hence

$$a = b_1 \vee \bigvee_{i=2}^{r} a_i$$

Continuing by induction, suppose

$$a = b_1 \vee b_2 \vee \cdots \vee b_l \vee \bigvee_{i=l+1}^{r} a_i$$

Then

$$b_{l+1} < b_1 \vee \ldots \vee b_l \vee \bigvee_{i=l+1} a_i$$

Then there are minimal subsets $I_{l+1} \subseteq \{l+1, \ldots, r\}$ and $J \subseteq \{1, \ldots, l\}$ such that

$$b_{l+1} < \bigvee_{j \in J} b_j \vee \bigvee \{a_i : i \in I_{l+1}\}$$

Without loss of generality assume $l+1 \in I_{l+1}$. It follows from minimality that $\{b_j, a_i : j \in J, i \in I_{l+1}\}$ are independent. Then

$$b_{l+1} \nleq \bigvee_{j \in J} b_j \vee \bigvee \{a_i : i \in I_{l+1} - \{l+1\}\}$$

and from Corollary 3.2 $\{b_{l+1}, b_j, a_i : j \in J, i \in I_{l+1} - \{l+1\}\}$ are independent. Again by Lemma 3.1,

$$a_{l+1} < b_{l+1} \vee \bigvee_{i \in J} b_j \vee \bigvee \{a_i : i \in I_{l+1} - \{l+1\}\}$$

Hence

$$a = b_1 \vee \ldots \vee b_{l+1} \vee \bigvee_{i=l+2}^{r} a_i$$

By induction we then find that

$$a = \bigvee_{i=1}^{r} b_i$$

But then

$$b_{r+1} < \bigvee_{i=1}^{r} b_i$$

which is a contradiction. Hence $b_1, \ldots, b_n$ are not independent.

Corollary 3.4

If $\{a_i : i = 1, \ldots, r\}$ and $\{b_i : i = 1, \ldots, n\}$ are bases for a, then $r = n$.

If $a \in L$ has a basis $a_1, \ldots, a_n$, then n is the *dimension* of a and denoted by $d(a) = n$. If a has a basis we say that a is *finite dimensional*. A set of atoms $a_i < a, i = 1, \ldots, n$ is a *maximal set* of independent atoms if they are independent and not in a larger set of independent atoms in a. If $a \leqslant b$ we use the notation $b - a = b \wedge a'$.

Lemma 3.5

If $a_i < a$, $i = 1, \ldots, r$ is a maximal set of independent atoms, then a_i, $i = 1, \ldots, r$, is a basis for a.

Proof: Suppose

$$\bigvee_{i=1}^{r} a_i < a$$

Then there is an atom

$$b \in a - \bigvee_{i=1}^{r} a_i$$

Then

$$b \nleq \bigvee_{i=1}^{r} a_i$$

and by Corollary 3.2 $\{b, a_i : i = 1, \ldots, r\}$ are independent which is a contradiction.

Corollary 3.6

If

$$a = \bigvee_{i=1}^{r} a_i$$

where the a_i's are atoms, then a is finite dimensional.

Proof: Find a subset of $\{a_i : i = 1, \ldots, n\}$ which forms a basis for a.

Corollary 3.7

If $d(a) = n$ and $a_i, i = 1, \ldots, n$, are independent atoms, then $\{a_i : i = 1, \ldots, n\}$ is a basis for a.

Proof: If $a_1, \ldots, a_n, b$ are independent for some atom $b \leqslant a$ we get a contradiction to Lemma 3.3. Therefore, $a_1, \ldots, a_n$ is a maximal set of independent atoms and by Lemma 3.5 must be a basis.

Corollary 3.8

If $a < b$ and $d(a) = d(b)$, then $a = b$.

Proof: Let $a_1, \ldots, a_n$ be a basis for a. Then by corollary 3.7, $a_1, \ldots, a_n$ is a basis for b, and hence $a = b$.

Corollary 3.9

If $a < b$ and b is finite dimensional, then $d(a) \leqslant d(b)$.

Proof: Suppose $r = d(b)$ and $a_1, \ldots, a_n$ are independent atoms in a, where $n > r$. This would contradict Lemma 3.3. Hence every maximal set of independent atoms in a has at most r elements. Applying Lemma 3.5, $d(a) < r$.

Recall that a *dimension function* d on a lattice $\mathscr{L}$ is a real valued function on $\mathscr{L}$ with the properties: (i) $d(0) = 0$, $d(a) \geqslant 0$, for all $a \in \mathscr{L}$; (ii) if $a < b$ and $a \neq b$, then $d(a) < d(b)$; (iii) $d(a \vee b) + d(a \wedge b) = d(a) + d(b)$ for all $a, b \in \mathscr{L}$.

Theorem 3.10

Let $c \in L$ be finite dimensional. Then d is a dimension function on $[0, c]$.

Proof: Property (i) is trivial, and (ii) follows from our previous corollaries. We now prove property (iii). Suppose $a, b \in [0, c]$ and $a \wedge b = 0$. Let $\{a_i\}$ be a basis for a and $\{b_j\}$ a basis for b. Then $a \vee b = \bigvee a_i \vee \bigvee b_j$. Now suppose $\{a_i, b_j\}$ are not independent. Then there is a b_k, say, such that

$$b_k < \bigvee a_i \vee \bigvee_{j \neq k} b_j$$

There are minimal index sets I, J such that

$$b_k \leqslant \bigvee_{i \in I} a_i \vee \bigvee_{j \in J} b_j$$

Now from the minimality of I and J we have that $\{a_i, b_j : i \in I, j \in J\}$ are independent and by the minimal superposition postulate

$$\left(\bigvee_{i \in I} a_i\right) \wedge (\vee \{b_j : j \in J \cup \{k\}\}) \neq 0$$

This contradicts $a \wedge b = 0$, so $\{a_i, b_j\}$ are independent and hence form a basis for $a \vee b$. Thus $d(a \vee b) + d(a \wedge b) = d(a) + d(b)$ in this case. In the general case we have

$$a \vee b = [(a - a \wedge b) \vee (a \wedge b)] \vee [(b - a \wedge b) \vee (a \wedge b)]$$
$$= (a - a \wedge b) \vee (a \wedge b) \vee (b - a \wedge b)$$

Now

$$(a - a \wedge b) \wedge (b - a \vee b) = [a \wedge (a \wedge b)'] \wedge [b \wedge (a \wedge b)'] = (a \wedge b) \wedge (a \wedge b)' = 0$$

Thus by our previous work

$$d(a \vee b) = d(a - a \wedge b) + d(a \wedge b) + d(b - a \wedge b) = d(a) + d(b) - d(a \wedge b)$$

Since modularity and the existence of a dimension function are equivalent (Varadarajan, 1968) we have the following corollary.

Corollary 3.11

If $a \in L$ is finite dimensional, then $[0, a]$ is a modular lattice.

4. *Conclusions*

We say that (L, M) is *completely irreducible* if for any interval $[0, a] \subseteq L$ the only elements of $[0, a]$ which are compatible with all elements of $[0, a]$ are 0 and a. This corresponds physically to the fact that there are no superselection rules in $[0, a]$ for all $a \in L$. It follows (cf. Varadarajan, 1968, Lemma 2.10) if (L, M) is completely irreducible and satisfies the assumptions of the previous section, that $[0, a]$ is a geometry for any finite dimensional element $a \in L$. We now have the following representation theorem.

Theorem 4.1

Let (L, M) be a completely irreducible quantum logic which satisfies the superposition principle, the minimal superposition postulate and suppose there is an $a \in L$ with $4 < d(a) < \infty$. Then there exists a division ring D, an involutive anti-automorphism θ of D, a vector space V over D, and a definite symmetric θ-bilinear form $\langle \cdot, \cdot \rangle$ on $V \times V$ such that L is isomorphic to the orthocomplemented lattice of all $\langle \cdot, \cdot \rangle$ closed subspaces of V.

In this theorem we have defined $\langle \cdot, \cdot \rangle$ closed subspaces in the following way. If T is a subspace we define $T^\perp = \{u \in V : \langle u, x \rangle = 0 \text{ for all } x \in T\}$ and T is $\langle \cdot, \cdot \rangle$ closed if $T = T^{\perp\perp}$. For the proof of this theorem the reader is referred to Varadarajan (1968, p. 179).

It is, of course, important to obtain more information about the division ring D. It is a classical result that if D has certain regularity properties (which it must in physical situations) then D is either the reals R, the complexes C, or the quaternions Q. For example, it is proved by Pontryagin (1932) that if D is a connected locally compact division ring in which addition and multiplication form topological groups, then D is R, C or Q. It can be shown (Varadarajan, 1968) that the division ring D in Theorem 4.1 is unique up to isomorphism. We call this essentially unique division ring the division ring *associated* with (L, M).

Theorem 4.2

Let (L, M) be a quantum logic satisfying the hypotheses of Theorem 4.1 and also the following two conditions: (i) the division ring D associated with (L, M) is R, C or Q; (ii) if $a, b \in L$, $a \neq 0$, $\neq 1$ and b is an atom, then there are atoms $b_1, b_2 \in L$ such that $b_1 \leqslant a$, $b_2 \leqslant a'$ and $b \leqslant b_1 \leqslant b_2$. Then L is isomorphic to the set of all closed subspace of a Hilbert space over D.

It is clear that the conditions given above are also necessary. For a proof of this theorem see Varadarajan (1968, Theorem 7.44).

5. *Remarks*

The hypotheses of Piron's theorem are stated differently by Piron (1964) than by Varadarajan (1968). The main difference is that Axiom (i) of Section 1 is replaced by the *covering law*: if a is an atom and $b \leqslant c \leqslant b \vee a$, then $c = b$ or $c = b \vee a$. Piron shows that the covering law implies Axiom (i). However, the author has seen no phenomenological justification for the covering law, so it seems to have no advantage over Axiom (i).† In any case our minimal superposition postulate can be used as an alternative to the covering law. A closely related axiom is the *semi-modularity*, which states that if (a,b) is a modular pair then so is (b,a). Pool (1968b) has given a physical justification for this axiom, but again his justification relies on addition axioms which are questionable.

One can note that our map $S \rightarrow S^-$ on subsets of P is a closure operation and that our minimal superposition postulate is closely related to the MacLane–Steinitz exchange axiom (Crapo & Rota, 1968). Thus the theory presented here is closely related to the theory of combinatorial geometries.

Finally, we would like to mention that the axiom $L_{m_1} \subseteq L_{m_2}$ implies $m_1 = m_2$ is not essential for this theory, and is included only to avoid certain minor technicalities. It is used only to prove that $\mathscr{M}$ is atomic. However, it can be shown that if this axiom does not hold $\mathscr{M}$ can be embedded in a unique smallest atomic lattice.

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† Added in proof: In a recent paper, work has been done on this by J. Jauch and C. Piron (1969). On the structure of Quantal Proposition Systems. *Helvetica Physica acta*, **42**, 842.

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